



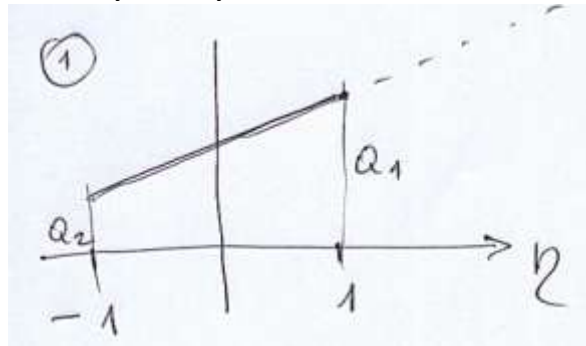
Finite element method (FEM1)

Lecture 5B. Numerical integration - examples

03.2025

Example 1. 1D case

Analytically:



$$f(\eta) = \frac{a_1 - a_2}{2} \cdot \eta + \frac{a_1 + a_2}{2}$$

$$\int_{-1}^1 f(\eta) d\eta = \int_{-1}^1 \left[\frac{a_1 - a_2}{2} \cdot \eta + \frac{a_1 + a_2}{2} \right] d\eta =$$

$$= \frac{1}{2} \cdot \frac{a_1 - a_2}{2} \cdot \eta^2 \Big|_{-1}^1 + \frac{a_1 + a_2}{2} \eta \Big|_{-1}^1 = 0 + \frac{a_1 + a_2}{2} (1 - (-1)) = a_1 + a_2$$

Numerical integration using 1 Gauss point:

$$\int_{-1}^1 f(\eta) d\eta \quad (\eta_1 = 0; w_1 = 2)$$

$$= f(0) \cdot 2 = \frac{a_1 + a_2}{2} \cdot 2 = a_1 + a_2$$

Polynomial degree	Number of Gauss points	ξ_i	w_i
1	1	0	2
3	2	$-\frac{1}{\sqrt{3}}$ $+\frac{1}{\sqrt{3}}$	1 1
5	3	$-\sqrt{0.6}$ 0 $+\sqrt{0.6}$	5/9 8/9 5/9

Example 2. 1D case

Analytically : $f(\eta) = 2\eta^2 - 3$

$$\int_{-1}^1 f(\eta) d\eta = \int_{-1}^1 (2\eta^2 - 3) d\eta = \left. \frac{2}{3} \eta^3 \right|_{-1}^1 - 3\eta \Big|_{-1}^1 =$$

$$= \frac{2}{3} \cdot 2 - 3 \cdot 2 = \frac{4}{3} - 6 = -4\frac{2}{3}$$

Numerical integration using 2 Gauss points :

$$\int_{-1}^1 f(\eta) d\eta = \sum_{i=1}^2 w_i \cdot f(\eta_i)$$

$$= 1 \cdot f\left(-\frac{1}{\sqrt{3}}\right) + 1 \cdot f\left(\frac{1}{\sqrt{3}}\right) =$$

$$= 1 \cdot \left(2 \cdot \left(\frac{1}{\sqrt{3}}\right)^2 - 3\right) + 1 \cdot \left(2 \cdot \left(\frac{1}{\sqrt{3}}\right)^2 - 3\right) = \frac{2}{3} - 3 + \frac{2}{3} - 3 =$$

$$= \frac{4}{3} - 6 = -4\frac{2}{3}$$

Polynomial degree	Number of Gauss points	ξ_i	w_i
1	1	0	2
3	2	$-\frac{1}{\sqrt{3}}$ $+\frac{1}{\sqrt{3}}$	1 1
5	3	$-\sqrt{0.6}$ 0 $+\sqrt{0.6}$	5/9 8/9 5/9

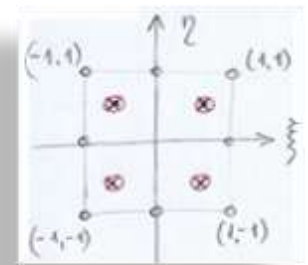
Example 3. 2D case

Analytically :

$$\int_{-1}^1 \int_{-1}^1 (\xi^3 + \xi \eta + \eta^2) d\eta d\xi =$$

$$= \int_{-1}^1 \left(\xi^3 \eta + \frac{1}{2} \xi \eta^2 + \frac{1}{3} \eta^3 \right) \Big|_{-1}^1 d\xi = \int_{-1}^1 \left(2\xi^3 + 0 + \frac{2}{3} \right) d\xi =$$

$$= \left(\frac{2}{4} \xi^4 + \frac{2}{3} \xi \right) \Big|_{-1}^1 = 0 + \frac{2}{3} \cdot 2 = \frac{4}{3}$$



Numerical integration using 2x2 Gauss points :

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta = \sum_{i=1}^2 \sum_{j=1}^2 w_i w_j f(\xi_i, \eta_j) \stackrel{+1/\sqrt{3}}{=}$$

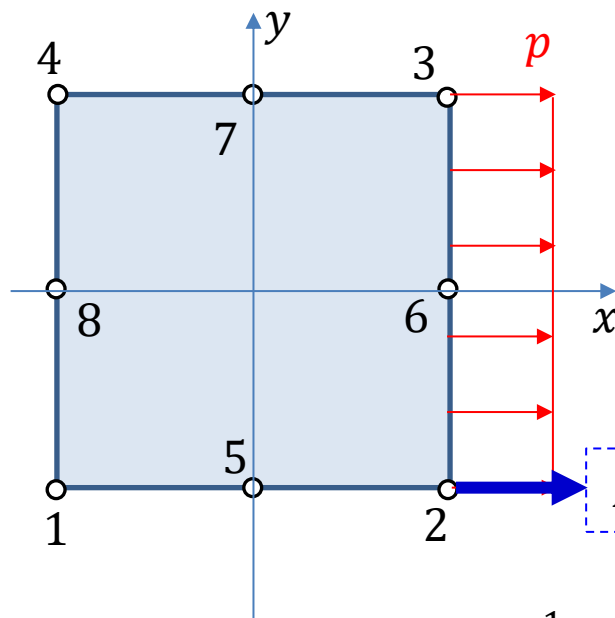
Polynomial degree	Num of GP	ξ_i	w_i
1	1	0	2
3	2	$-1/\sqrt{3}$ $+1/\sqrt{3}$	1 1

$$= 1 \cdot 1 \cdot f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + 1 \cdot 1 \cdot f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + 1 \cdot 1 \cdot f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + 1 \cdot 1 \cdot f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) =$$

$$= \left[\left(-\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{3} + \frac{1}{3} \right] + \left[\left(-\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{3} + \frac{1}{3} \right] + \left[\left(\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{3} + \frac{1}{3} \right] + \left[\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{3} + \frac{1}{3} \right] =$$

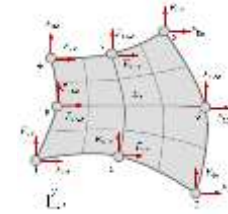
$$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

Example 4. Determination of equivalent forces in an 8-node element (analytical vs numerical integration)



equivalent load vector due to surface load:

$$[F^p]_e = t_e \int_0^l [p][N] ds$$



$$dy = \frac{l}{2} d\eta$$

$$F_{3e}^p = t_e \int_0^l p N_2 dy$$

$$N_2(1, \eta) = -\frac{1}{2}(1 - \eta)\eta$$

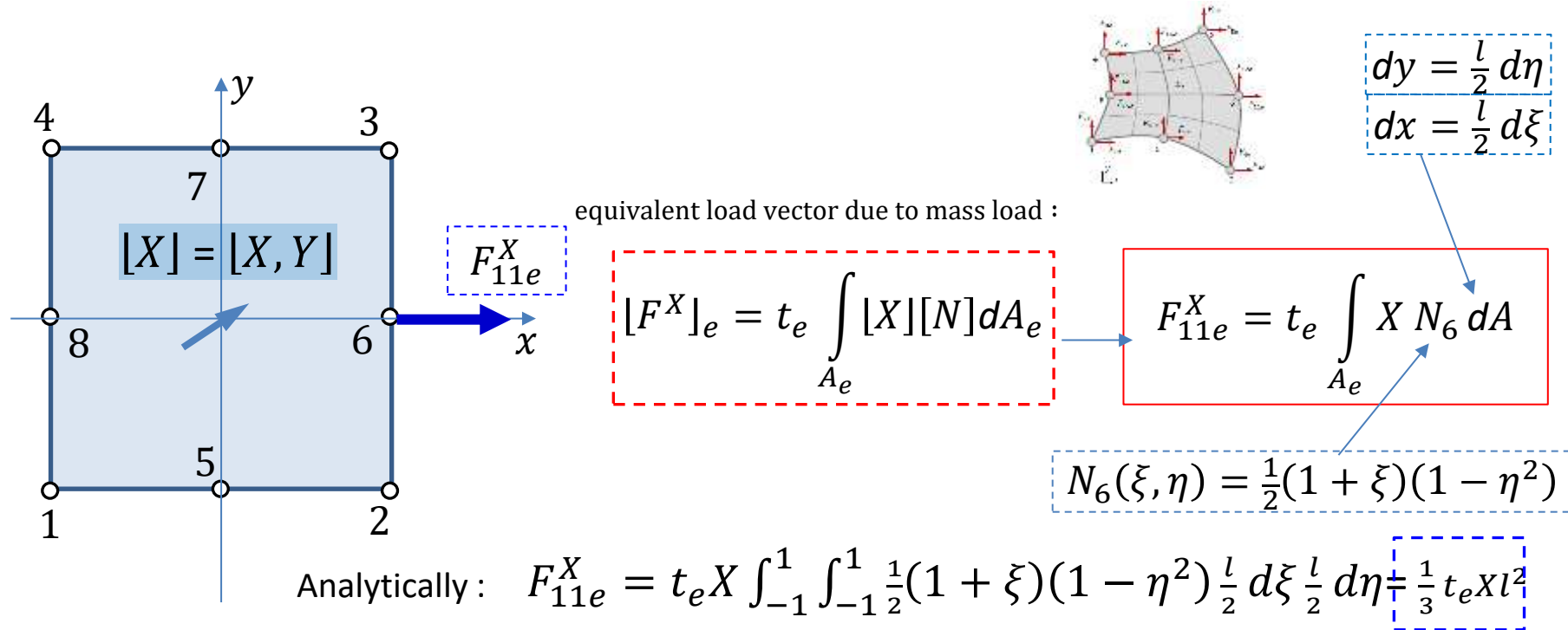
Analytically:
$$F_{3e}^p = t_e p \int_{-1}^1 \left[\frac{1}{2}(\eta - 1)\eta \right] \frac{l}{2} d\eta = \frac{pl}{4} t_e \left(\frac{1}{3}\eta^3 - \frac{1}{2}\eta^2 \right) \Big|_{-1}^1 = \frac{pl}{6} t_e$$

Numerical integration using 2 Gauss points:

for 2 Gauss points: $\eta_1 = -\frac{1}{\sqrt{3}}; \eta_2 = \frac{1}{\sqrt{3}}; w_1 = w_2 = 1$

$$\begin{aligned} F_{3e}^p &= \frac{1}{4} t_e pl \int_{-1}^1 (\eta - 1)\eta d\eta = \frac{1}{4} t_e pl \left[1 \cdot \left(-\frac{1}{\sqrt{3}} - 1 \right) \left(-\frac{1}{\sqrt{3}} \right) + 1 \cdot \left(\frac{1}{\sqrt{3}} - 1 \right) \left(\frac{1}{\sqrt{3}} \right) \right] = \\ &= \frac{1}{4} t_e pl \left[\frac{1}{3} + \frac{1}{\sqrt{3}} + \frac{1}{3} - \frac{1}{\sqrt{3}} \right] = \frac{pl}{6} t_e \end{aligned}$$

Example 5. Determination of equivalent forces in an 8-node element (analytical vs numerical integration)



equivalent load vector due to mass load :

$$[F^X]_e = t_e \int_{A_e} [X][N] dA_e$$

$$F_{11e}^X = t_e \int_{A_e} X N_6 dA$$

$$N_6(\xi, \eta) = \frac{1}{2}(1 + \xi)(1 - \eta^2)$$

Analytically : $F_{11e}^X = t_e X \int_{-1}^1 \int_{-1}^1 \frac{1}{2}(1 + \xi)(1 - \eta^2) \frac{l}{2} d\xi \frac{l}{2} d\eta = \frac{1}{3} t_e X l^2$

Numerical integration using 2x2 Gauss points :

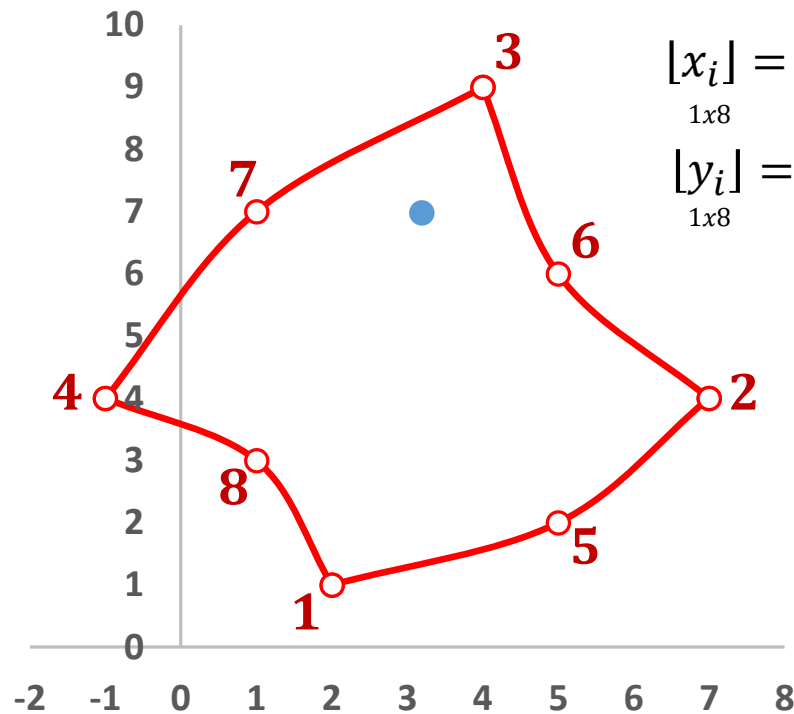
for 2 Gauss points: $\xi_1 = \eta_1 = -\frac{1}{\sqrt{3}}$; $\xi_2 = \eta_2 = \frac{1}{\sqrt{3}}$; $w_1 = w_2 = 1$

$$F_{11e}^X = t_e \frac{l^2}{8} X \int_{-1}^1 \int_{-1}^1 (1 + \xi)(1 - \eta^2) d\xi d\eta =$$

$$= t_e \frac{l^2}{8} X \left[1 \cdot \left(1 - \frac{1}{\sqrt{3}}\right) \left(1 - \left(-\frac{1}{\sqrt{3}}\right)^2\right) + 1 \cdot \left(1 + \frac{1}{\sqrt{3}}\right) \left(1 - \left(-\frac{1}{\sqrt{3}}\right)^2\right) + \right.$$

$$\left. + 1 \cdot \left(1 - \frac{1}{\sqrt{3}}\right) \left(1 - \left(\frac{1}{\sqrt{3}}\right)^2\right) + 1 \cdot \left(1 + \frac{1}{\sqrt{3}}\right) \left(1 - \left(\frac{1}{\sqrt{3}}\right)^2\right) \right] = \frac{1}{3} t_e X l^2$$

Example 6. 8-node element - numerical integration ($n=3$).
Find the area of the element and the volume V_0 .

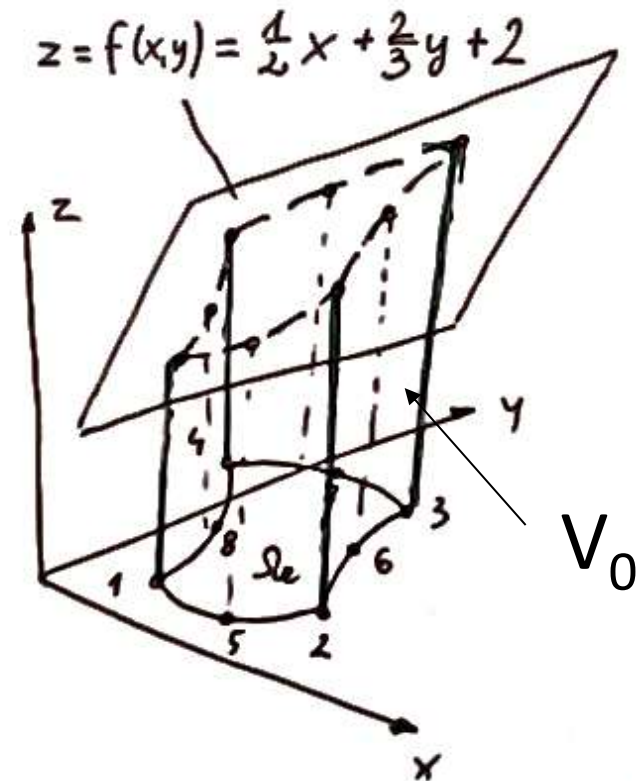


$$[x_i] = [2, 7, 4, -1, 5, 5, 1, 1];$$

1×8

$$[y_i] = [1, 4, 9, 4, 2, 6, 7, 3]$$

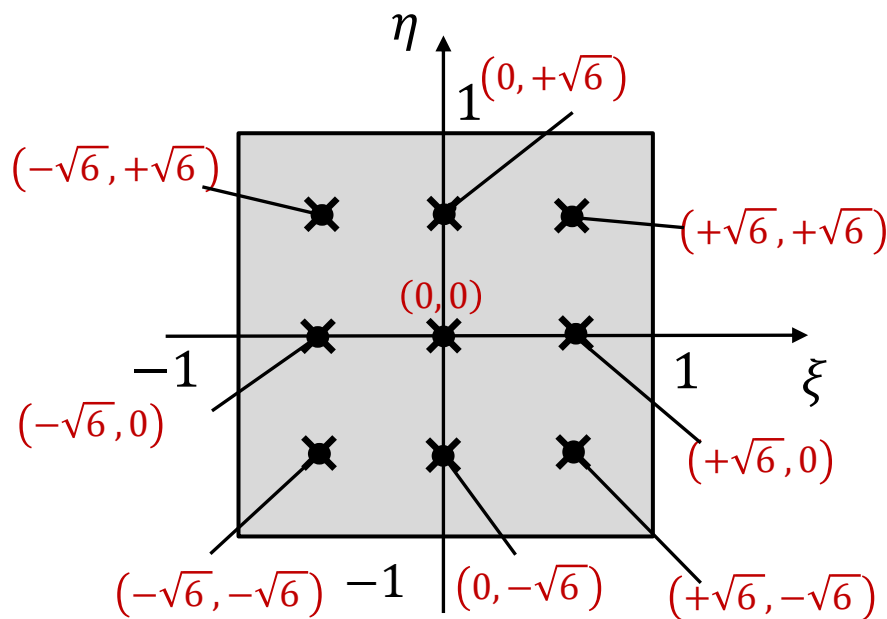
1×8



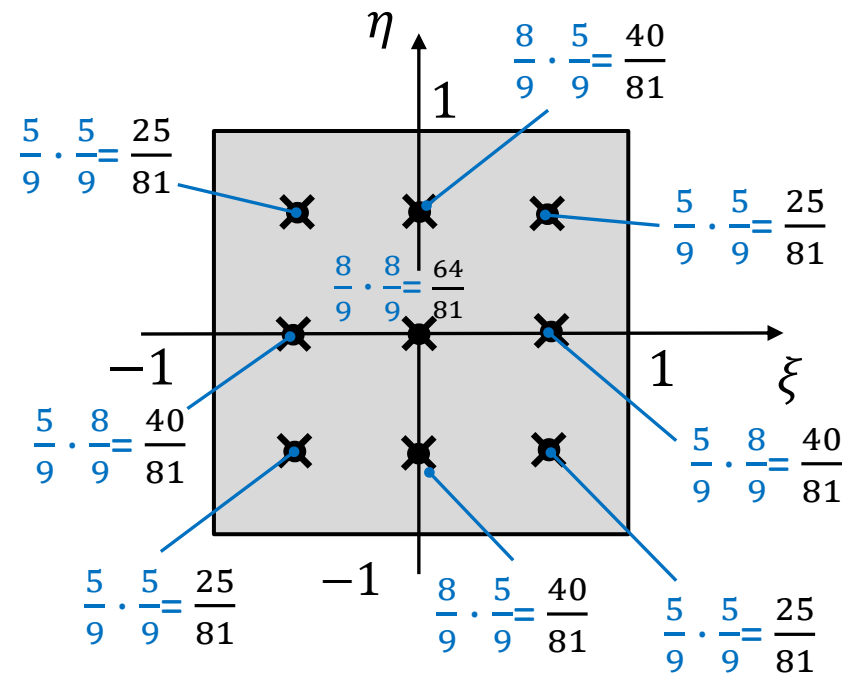
Reminder: Gaussian quadrature rule for 2D elements

$$n = 3 : \left[\xi_1 = \eta_1 = -\sqrt{0.6}, \quad \xi_2 = \eta_2 = 0, \quad \xi_3 = \eta_3 = \sqrt{0.6} \right]$$

$$w_1 = w_3 = \frac{5}{9} \quad ; \quad w_2 = \frac{8}{9}$$



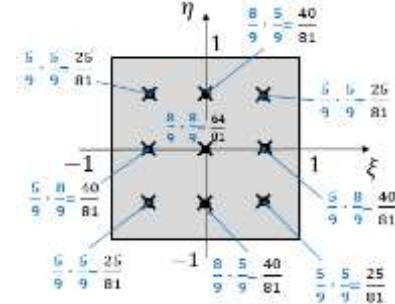
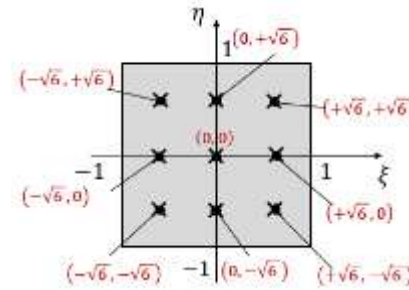
$$(\xi_i, \eta_j)$$



$$w_i w_j$$

Volume calculation

$$x = [N(\xi, \eta)]_{1 \times 8} \{x_i\}_e \quad ; \quad y = [N(\xi, \eta)]_{1 \times 8} \{y_i\}_e$$



$$V_0 = \iint_A f(x, y) dx dy =$$

$$= \int_{-1}^1 \int_{-1}^1 \left(\frac{1}{2}x + \frac{2}{3}y + 2 \right) \det[J(\xi, \eta)] d\xi d\eta =$$

$$= \int_{-1}^1 \int_{-1}^1 \left(\frac{1}{2} [N(\xi, \eta)] \{x_i\} + \frac{2}{3} [N(\xi, \eta)] \{y_i\} + 2 \right) \det[J(\xi, \eta)] d\xi d\eta =$$

$$= \left| \det[J] = \left(\frac{\partial [N(\xi, \eta)]}{\partial \xi} \{x_i\}_e \right) \left(\frac{\partial [N(\xi, \eta)]}{\partial \eta} \{y_i\}_e \right) - \left(\frac{\partial [N(\xi, \eta)]}{\partial \xi} \{y_i\}_e \right) \left(\frac{\partial [N(\xi, \eta)]}{\partial \eta} \{x_i\}_e \right) \right| =$$

The calculation of this determinant was shown in the presentation:
Lecture 04B

$$= \left(\frac{1}{2} [N(-\sqrt{6}, -\sqrt{6})] \{x_i\} + \frac{2}{3} [N(-\sqrt{6}, -\sqrt{6})] \{y_i\} + 2 \right) \det [J(-\sqrt{6}, -\sqrt{6})] \cdot \frac{5}{9} \cdot \frac{5}{9} +$$

$$+ \left(\frac{1}{2} [N(0, -\sqrt{6})] \{x_i\} + \frac{2}{3} [N(0, -\sqrt{6})] \{y_i\} + 2 \right) \det [J(0, -\sqrt{6})] \cdot \frac{8}{9} \cdot \frac{5}{9} +$$

$$+ \left(\frac{1}{2} [N(\sqrt{6}, -\sqrt{6})] \{x_i\} + \frac{2}{3} [N(\sqrt{6}, -\sqrt{6})] \{y_i\} + 2 \right) \det [J(\sqrt{6}, -\sqrt{6})] \cdot \frac{5}{9} \cdot \frac{5}{9} +$$

$$+ \left(\frac{1}{2} [N(-\sqrt{6}, 0)] \{x_i\} + \frac{2}{3} [N(-\sqrt{6}, 0)] \{y_i\} + 2 \right) \det [J(-\sqrt{6}, 0)] \cdot \frac{8}{9} \cdot \frac{5}{9} +$$

$$+ \left(\frac{1}{2} [N(0, 0)] \{x_i\} + \frac{2}{3} [N(0, 0)] \{y_i\} + 2 \right) \det [J(0, 0)] \cdot \frac{8}{9} \cdot \frac{8}{9} + \dots = 220.4 \text{ mm}^3$$

Area calculation

$$A = \iint_A dx dy =$$

$$= \int_{-1}^1 \int_{-1}^1 \det[J(\xi, \eta)] d\xi d\eta =$$

$$\begin{aligned} &= \det[J(-\sqrt{6}, -\sqrt{6})] \cdot \frac{5}{9} \cdot \frac{5}{9} + \det[J(0, -\sqrt{6})] \cdot \frac{8}{9} \cdot \frac{5}{9} + \det[J(\sqrt{6}, -\sqrt{6})] \cdot \frac{5}{9} \cdot \frac{5}{9} + \\ &+ \det[J(-\sqrt{6}, 0)] \cdot \frac{8}{9} \cdot \frac{5}{9} + \det[J(0, 0)] \cdot \frac{8}{9} \cdot \frac{8}{9} + \det[J(0, \sqrt{6})] \cdot \frac{8}{9} \cdot \frac{5}{9} + \\ &+ \det[J(-\sqrt{6}, \sqrt{6})] \cdot \frac{5}{9} \cdot \frac{5}{9} + \det[J(0, \sqrt{6})] \cdot \frac{8}{9} \cdot \frac{5}{9} + \det[J(\sqrt{6}, \sqrt{6})] \cdot \frac{5}{9} \cdot \frac{5}{9} = 33\,333 \text{ mm}^2 \end{aligned}$$

